Table 1 Various solutions of Eq. (14) for specific cases of temperature variations ($d\nu/dt = 0$)

T	T_e
$dT/dt = 0$	$T(1 - e^{-\nu\beta t}) + T_0 e^{-\nu\beta t}$ (also developed in Ref. 3)
$T_{0g}(1 + at)$	$T_{0g}(1 + at - a/\nu\beta) + e^{-\nu\beta t} \times [T_0 - T_{0g}(1 - a/\nu\beta)]$
$T_{0g} + T_a e^{at}$	$T_{0g} + T_a[\nu\beta/(a + \nu\beta)]e^{at} + \{T_0 - T_{0g} - T_a[\nu\beta/(a + \nu\beta)]\}e^{-\nu\beta t}$
$T_{0g} + T_a \sin at$	$T_{0g} + T_a[\beta\nu/(a^2 + \beta^2\nu^2)](\beta\nu \sin at - a \cos at) + \{T_0 - T_{0g} + T_a[a\beta\nu/(a^2 + \beta^2\nu^2)]\}e^{-\beta\nu t}$

$$B^{-1} = \int \frac{4kT\nu}{M} \left\{\exp\left[\left(\frac{2m}{M}\right) \int \nu dt'\right]\right\} dt \qquad (12)$$

and the general solution of A is then written as

$$A = \frac{\exp[(2m/M)\int \nu dt]}{\int (4\,kT\nu/M)[\exp\{(2m/M)\int \nu dt'\}]\,dt} \qquad (13)$$

Since the electron temperature T_e is just $m/2kA$, Eq.(13) gives

$$T_e = \exp(-\beta \int \nu dt)\ \{\int \beta\ T\nu \times [\exp(\beta \int \nu dt')]\,dt + K\} \qquad (14)$$

where $\beta = 2m/M$ and K is the integration constant.

Closed Form Solutions for T_e

In Table 1 are given various solutions of Eq. (14) for some specific cases of temperature variations. The initial electron temperature in Table 1 is T_0.

Approximate Solution of the General Problem

It would be desirable to have a solution to the general case in which both the gas temperature and density can have quite arbitrary temporal variations. Since any reasonable arbitrary temperature variation may be divided into intervals, and the ith interval approximated by $T_i = T_{0gi}\,(1 + a_i t)$, and similarly for the collision frequency, it is worthwhile to find a solution for this case which may be used successively on each interval to give an approximate solution to the over-all problem.

Integrating by parts, the integral in the braces of Eq. (14) becomes

$$\beta \int T\nu \exp(\beta \int \nu dt')\ dt = T \exp(\beta \int \nu dt) - \int \exp(\beta \int \nu dt')\ (dT/dt)\ dt \qquad (15)$$

Making the substitutions $T = T_{0g}\,(1 + at)$, $\nu = \nu_0\,(1 + bt)$, Eq. (15) becomes

$$\begin{aligned}\beta \int T\nu(\beta \int \nu dt')dt = T_{0g}\,(1 + at) \times \\ \exp[\beta\nu_0 t\,(1 + bt/2)] - aT_{0g} \int \exp[\beta\nu_0 t\,(1 + bt/2)] \times \\ dt = T_{0g}\,(1 + at) \exp[\beta\nu_0 t\,(1 + bt/2)] + \\ [aT_{0g}/(\beta b\nu_0)^{1/2}] \exp[-\beta\nu_0/2b] \operatorname{erf}\{(2\beta\nu_0)^{1/2} \times \\ [t(b/2)^{1/2} + 1/(2b)^{1/2}]\} + C\end{aligned} \qquad (16)$$

where erf (x) denotes the error function.[2] Combining Eqs. (16) and (14) gives the final solution as

$$\begin{aligned}T_e = T_{0g}\,(1 + at) + [aT_{0g}/(\beta\nu_0 b)^{1/2}] \times \\ \exp[-\beta\nu_0\,(t + bt^2/2 + 1/2b)] \operatorname{erf}\{(2\beta\nu_0)^{1/2} \times \\ [t(b/2)^{1/2} + 1/(2b)^{1/2}]\} + \{T_0 - T_{0g} \times \\ (1 + [a/(\beta\nu_0 b)^{1/2}] \exp(-\beta\nu_0/2b) \operatorname{erf}(\beta\nu_0/2)^{1/2})\} \times \\ \exp[-\beta\nu_0 t\,(1 + bt/2)]\end{aligned}$$

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Aerodynamic Pitching Derivatives of a Wedge in Hypersonic Flow

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Introduction

In a recent series of experiments carried out in the University of Southampton gun tunnel, East[1] has measured the aerodynamic stiffness and damping derivatives m_θ and $m_{\dot\theta}$, respectively [see Eqs. (20–22) for definitions], at a freestream Mach number $M_\infty \doteqdot 9.7$ of a two-dimensional wedge aerofoil oscillating in pitch about several positions along its chord; the experimental results for the sharp leading edge model are illustrated in Fig. 1. Since the wedge half angle δ was equal to $9\frac{1}{2}°$ giving a thickness parameter $M_\infty\delta = 1.6$, he compared his measured values of m_θ and $m_{\dot\theta}$ with theoretical predictions that embodied the "strong-shock" piston theory suggested by Miles.[2] The agreement obtained between the experimental results and the theory was certainly more than qualitative. Within the limits of experimental scatter, the measured values of m_θ were well correlated by the theory, although the experimental values of $m_{\dot\theta}$ showed a certain skewness about the theoretical prediction which made them fall below the theoretical prediction for forward positions of the pivot point and above the theoretical prediction for aft positions of the pivot point. Indeed, the experimental results indicated positive values of the damping derivative for positions of the pivot between 0.2 and 0.4 chord. East attributed the discrepancies between experiment and theory primarily to the interference of the flow along the two sides of the two-dimensional model which was caused by laminar

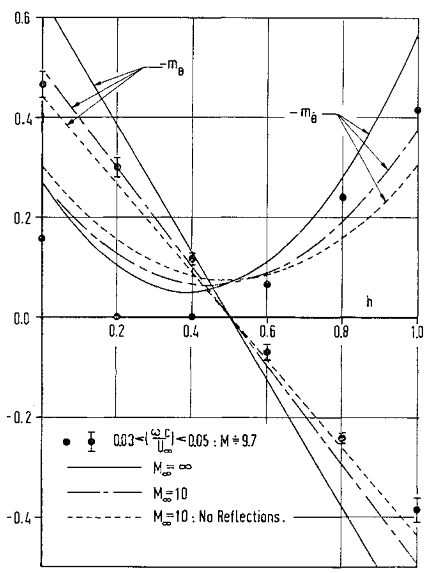

Fig. 1 Variation of the aerodynamic stiffness and damping derivatives with pivot position.

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boundary-layer separation on the supporting side walls. Schlieren photographs showed that between 15 and 35% (depending on the pivot position along the wedge) of the models' surface was within the influence of the shock produced by the boundary-layer flow over the side walls.

However, under East's experimental conditions, the influence of the secondary unsteady reflected waves from the bow shock wave back onto the wedge could have given rise to some degree of skewness in the measured values of $m_{\dot{\theta}}$, since the unsteady pressure increments due to these reflections would not, in general, be in phase with the unsteady normal component of the velocity of the wedge. Miles' strong shock piston theory neglects these reflections, although for strong shocks their strength would not be negligible by comparison with that of the incident waves. The following analysis was carried out in an attempt to estimate the effect of the reflected waves on the values of m_θ and $m_{\dot{\theta}}$ for small values of the frequency parameter $(\omega c/U_\infty)$.

Theory

We use the piston theory model for unsteady two-dimensional hypersonic flow over a slender body which likens the motion of the fluid in planes transverse to the freestream direction to the motion produced ahead of a piston moving in a uniform tube (see Fig. 2). If we follow the motion of a fluid slab in region 2 between the shock and the surface of the model as it moves in the x direction with velocity U_2, the equations that govern the flow in the transverse y direction become

$$\frac{\partial \rho}{\partial \tau} + \frac{\partial}{\partial y}(\rho v) = 0 \tag{1}$$

$$\rho\left(\frac{\partial v}{\partial \tau} + v\frac{\partial v}{\partial y}\right) + \frac{\partial p}{\partial y} = 0 \tag{2}$$

$$\left(\frac{\partial}{\partial \tau} + v\frac{\partial}{\partial y}\right)\ln\left(\frac{p}{\rho^\gamma}\right) = 0 \tag{3}$$

where ρ is the density, p is the pressure, v is the velocity component in the y direction, and τ is the time which is measured from the instant that the fluid slab reaches the leading edge of the wedge. The velocity of the equivalent piston for a wedge that oscillates periodically with a constant amplitude and frequency ω about a pivot distance hc from the leading edge is

$$v_p = U_2\delta + \dot{\theta}(x - hc) + \theta U_2 \tag{4}$$

where $\theta = \theta_0 \exp(i\omega t)$ (real part) and t is the time measured from some arbitrary datum. The term $U_2\delta$ corresponds to the steady piston velocity, whereas the remaining terms correspond to the unsteady oscillation; it will be assumed that their magnitude is much less than $U_2\delta$, and their sum will be regarded as the unsteady perturbation to the piston velocity. Thus, as we follow the motion of the fluid slab over the wedge,

$$\bar{v}_p = \theta_0[i\omega(U_2\tau - hc) + U_2]\exp[i(\omega\tau + \phi)] \tag{5}$$

where the bar indicates a perturbation quantity, and $\phi = \omega(t - \tau)$ is a constant for each fluid slab.

The conservation equations (1–3) may be linearized by writing $p = p_2 + \bar{p}$, $\rho = \rho_2 + \bar{\rho}$, and $v = v_2 + \bar{v}$ and then rearranged to give

$$\left[\frac{\partial}{\partial \tau} + (v_2 + a_2)\frac{\partial}{\partial y}\right](\bar{p} + \rho_2 a_2 \bar{v}) = 0 \tag{6}$$

$$\left[\frac{\partial}{\partial \tau} + (v_2 - a_2)\frac{\partial}{\partial y}\right](\bar{p} - \rho_2 a_2 \bar{v}) = 0 \tag{7}$$

where

$$a_2^2 = \gamma p_2/\rho_2$$

and

$$(\bar{p}/p_2) + \gamma(\bar{\rho}/\rho_2) = \text{const} \tag{8}$$

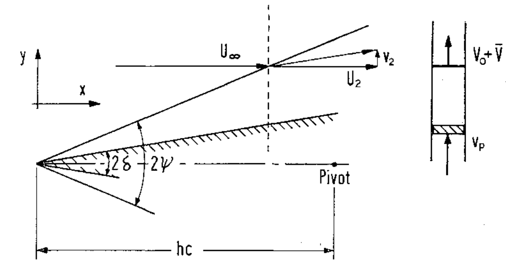

Fig. 2 Model geometry and coordinate system.

for each fluid element within the slab. Equations (6) and (7) have solutions of the form

$$\bar{p}/\rho_2 a_2 v_2 = \tfrac{1}{2}[f(\xi) + g(\eta)] \tag{9}$$

$$\bar{v}/v_2 = \tfrac{1}{2}[f(\xi) - g(\eta)] \tag{10}$$

where f and g are linear functions of the characteristic parameters $\xi = \tau(v_2 + a_2) - y$ and $\eta = \tau(v_2 - a_2) - y$. Both f and g may be obtained explicitly by satisfying the appropriate boundary conditions at the shock and at the piston face. We shall apply the boundary conditions using a method which closely follows that employed by Dem'yanov[3] and subsequently by Spence and Woods,[4] who investigated the unsteady perturbations caused by boundary-layer growth in shock-tube flow. Small perturbations to the flow quantities immediately behind the shock, which are due to the unsteady waves catching up with the shock, will give rise to perturbations $\bar{V}$ in the velocity of the shock from its steady velocity $V_0 = U_\infty\psi$, where ψ is the shock-wave angle. The linear expressions that relate these perturbation quantities are

$$\bar{v}_s/v_2 = \lambda\bar{V}/V_0 \tag{11}$$

$$\bar{p}_s/\rho_2 a_2 v_2 = 2\mu M\bar{V}/V_0 \tag{12}$$

where $\lambda = (M_0^2 + 1)/(M_0^2 - 1)$, $\mu = M_0^2/(M_0^2 - 1)$, $M_0 = V_0/a_\infty$, and $M = (V_0 - v_2)/a_2$, and the suffix s identifies the conditions immediately behind the shock. The expressions (11) and (12) are obtained in a straightforward manner by perturbing the shock wave Rankine Hugoniot relations. We now substitute (11) and (12) into Eqs. (10) and (9), respectively, and write the functions f and g in terms of the coordinates of the undisturbed shock: this last step being consistent with linear theory. Thus we have

$$2\mu M\bar{V}/V_0 = \tfrac{1}{2}[f(\alpha\tau) + g(\beta\tau)] \tag{13}$$

$$\lambda\bar{V}/V_0 = \tfrac{1}{2}[f(\alpha\tau) - g(\beta\tau)] \tag{14}$$

where $\xi/\tau = (v_2 + a_2 - V_0) = \alpha$ and $\eta/\tau = (v_2 - a_2 - V_0) = \beta$, at the shock. Therefore, in general,

$$\frac{\bar{p}}{\rho_2 a_2 v_2} = G\frac{\bar{V}}{V_0}(\xi) + H\frac{\bar{V}}{V_0}(\eta) \tag{15}$$

$$\frac{\bar{v}}{V_2} = G\frac{\bar{V}}{V_0}(\xi) - H\frac{\bar{V}}{V_0}(\eta) \tag{16}$$

where $G = (\mu M + \lambda/2)$ and $H = (\mu M - \lambda/2)$.

Now, at the piston, $\xi = a_2\tau$ and $\eta = -a_2\tau$; therefore, from Eqs. (5) and (16) we obtain

$$\frac{\theta_0 U_2}{v_2}\left[i\omega\left(\tau - \frac{hc}{U_2}\right)\right]\exp[i(\omega\tau + \phi)] = G\frac{\bar{V}}{V_0}(a_2\tau) - H\frac{\bar{V}}{V_0}(-a_2\tau) \tag{17}$$

Provided that $(\omega c/U_2) \ll 1$, we may seek a series solution to Eq. (17) in the form

$$\frac{\bar{V}}{V_0}(z) = \sum_{r=0}^{\infty} q_r z^r$$

where z is a general independent variable, by expanding the left-hand side of the equation in ascending powers of τ and then equating the coefficients of like terms from both sides of the equation. Having carried out this process, we find that the rth coefficient of the series solution for $\bar{V}/V_0$ is

$$q_r = \frac{(i\omega)^r}{r!}\frac{\theta_0 U_2}{v_2 a_2^r}\left[\frac{(1 - i\omega hc/U_2) + r}{G - (-1)^r H}\right]\exp(i\phi)$$

After some rearrangement, we are then able to obtain the following expression from (15) for the unsteady pressure perturbation at a general fixed point $x = U_2\tau$ on the surface of the wedge:

$$\bar{p}_p/\rho_2 a_2 v_2 = L(\theta_0 U_2/v_2)\{[1 - i\omega hc/U_2 + i\omega x/U_2][1 - N\exp(-2i\omega x/U_2)]\}\exp(i\omega t) \quad (18)$$

where

$$L = \frac{1 + (H/G)^2}{1 - (H/G)^2} \qquad N = \frac{2(H/G)}{1 + (H/G)^2}$$

Now

$$\exp\left(-\frac{2i\omega x}{U_2}\right) = 1 - \frac{2i\omega x}{U_2} + O\left[\left(\frac{\omega x}{U_2}\right)^n\right] \qquad n = 2, 3, \ldots$$

therefore,

$$\frac{\bar{p}_p}{\rho_2 a_2 v_2} \doteqdot L\frac{U_2}{v_2}\left\{\left[\theta + \dot{\theta}\left(\frac{x - hc}{U_2}\right)\right] - N\left[\theta - \dot{\theta}\left(\frac{x + hc}{U_2}\right)\right]\right\} \quad (19)$$

If we neglect the secondary wave reflections, then $L = 1$, $N = 0$, and Eq. (19) reduces to the simple plane wave relation used in previous applications of the piston theory (see, for example, Lighthill[5] and Miles[2]).

Now the aerodynamic stiffness is given by

$$-m_\theta = \frac{1}{\rho_\infty U_\infty^2 c^2}\left(-\frac{\partial M}{\partial \theta}\right)_{\theta=\dot{\theta}=0} \quad (20)$$

and the aerodynamic damping by

$$-m_{\dot{\theta}} = \frac{1}{\rho_\infty U_\infty c^3}\left(-\frac{\partial M}{\partial \dot{\theta}}\right)_{\theta=\dot{\theta}=0} \quad (21)$$

where

$$-M = 2\int_0^c (x - hc)\,\bar{p}_p dx \quad (22)$$

Therefore, from Eqs. (19–22) we obtain

$$-m_\theta = 2L(\rho_2 a_2 U_2/\rho_\infty U_\infty^2)[(\tfrac{1}{2} - h)(1 - N)] \quad (23)$$

$$-m_{\dot{\theta}} = 2L(\rho_2 a_2/\rho_\infty U_\infty)[(\tfrac{1}{3} - h + h^2) + N(\tfrac{1}{3} - h^2)] \quad (24)$$

For strong shocks, the coefficients in front of the square brackets in Eqs. (23) and (24) reduce to simple functions of γ and δ, since

$$\frac{U_2}{U_\infty} = 1 \qquad \frac{2\rho_2 a_2}{\rho_\infty U_\infty} = (\gamma + 1)\left(\frac{2\gamma}{\gamma - 1}\right)^{1/2}\delta$$

$$\lambda = \mu = 1 \qquad M = \left(\frac{\gamma - 1}{2\gamma}\right)^{1/2}$$

Concluding Remarks

In Fig. 1, East's experimental values of $-m_\theta$ and $-m_{\dot{\theta}}$ are compared with the theoretical predictions given by Eqs. (23) and (24), which include the effect of the secondary reflections, and also with the equivalent expressions that neglect the reflections, i.e., $L = 1$, $N = 0$. It can be seen that the agreement is generally improved when the reflections are taken into account and that their effect, even for moderate strength shock waves, is not insignificant. However, the value of the "reflection coefficient" N can never be large enough for $m_{\dot{\theta}}$ to be positive even when $M_\infty = \infty$. Therefore, it appears that the major part of the remaining discrepancy between theory and experiment is due to departures from the ideal two-dimensional flow over the model as suggested by East.

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Injection Thrust Termination and Modulation in Solid Rockets

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Nomenclature

A	=	area, in.^{2}
C^*	=	characteristic velocity, fps
C_D	=	discharge coefficient, dimensionless
h	=	enthalpy, Btu/lb
ΔH_v	=	latent heat of vaporization, Btu/lb
P	=	pressure, psia
r	=	burning rate, in./sec
R	=	gas constant, ft/°R
T	=	temperature, °R
V	=	volume, in.^{3}
W	=	weight, lb
y	=	weight fraction of vaporized water, dimensionless
Ω	=	defined in denominator of Eq. (6)
γ	=	specific heat ratio, dimensionless
ρ	=	density, lb/in.^{3}
θ	=	time, sec

Subscripts

c	=	chamber
g	=	propellant gas
p	=	solid propellant
s	=	steam
w	=	water
$(\ )'$	=	conditions after injection
1, 2	=	time in transient period

RECENT studies by the authors indicate that positive thrust termination and a degree of thrust modulation can be achieved in solid rocket motors by water injection.

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